

# Numerical Simulation of the Taylor-Green Vortex at $Re=1600$ with the Discontinuous Galerkin Spectral Element Method

Contribution to testcase 3.3 of the  
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## Abstract

In the following work, we present the results of selected simulations of the classical Taylor-Green vortex problem with a variant of the Discontinuous Galerkin method (DG) labeled the “Discontinuous Galerkin Spectral Element Method” (DGSEM). In the classical DGSEM formulation, the non-linear fluxes are co-located on the solution grid, leading to a highly efficient scheme but possible aliasing errors. Polynomial de-aliasing techniques proposed by Kirby and Karniadakis [5] avoid these errors, but incur a higher computational cost. We show results for the co-location and fully de-aliased versions of DGSEM, along with results for a locally adaptive de-aliasing approach.

## Taylor-Green Vortex flow

The Taylor-Green vortex flow problem constitutes the simplest flow for which a turbulent energy cascade can be observed numerically. Starting from an initial analytical solution containing only a single length scale, the flow field undergoes a rapid build-up of a fully turbulent dissipative spectrum because of non-linear interactions of the developing eddies (Fig. 1). The resulting flow field exhibits the features of an isotropic, homogeneous turbulence and is often used in code validation or evaluation of numerical approaches to subgrid scale modeling [2], [3], [4].

All our computations were run on a structured Cartesian grid of hexahedral elements, covering a triple-periodic box of size  $[-\pi, \pi]^3$ . The physical time frame from 0s to 20s was covered according to the problem description, starting from the initial analytical solution with given velocity and pressure fields, a constant temperature and an essentially incompressible flow field with a Mach number of  $Ma = 0.1$ .

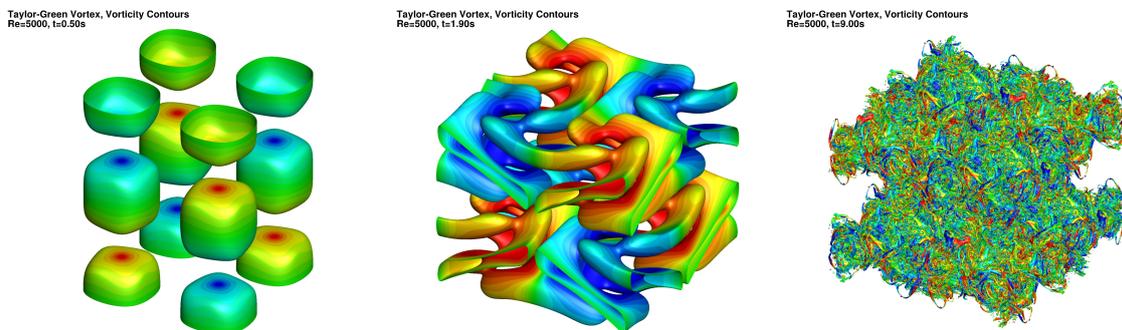


Figure 1: Taylor-Green Vortex ( $Re = 5000$ ). Isocontours of vorticity magnitude, colored by helicity at  $t = 0.5s, 1.9s$  and  $9.0s$

## Code Framework

Our code framework is based on a variant of the Discontinuous Galerkin method labeled the “Discontinuous Galerkin Spectral Element method”, see Kopriva [6], and solves the compressible Navier-Stokes equations. The implementation allows the selection of arbitrary polynomial order and thus enables us to study the features of high order formulations very efficiently within our framework. In addition, the integration precision of the flux terms can be chosen independently from the solution to allow polynomial de-aliasing. Explicit time integration is achieved by a 5-stage 4th order Runge-Kutta scheme.

The code is accompanied by a postprocessing tool for visualization and a-posteriori extraction of relevant flow features and a 3D Fast Fourier transform for the analysis of flow spectra. The whole framework is fully MPI-parallelized, where special care has been taken to achieve a high parallel efficiency and excellent scaling. On the JuQueen (IBM BlueGene/Q system, Jülich Supercomputing Center) system, a perfect strong scaling was measured on up to 216,000 ranks.

In this work, we present computations performed Cray XC40 “Hornet” cluster (TauBench of 4.7s) at the High Performance Computing Center Stuttgart (HLRS) on up to 6912 cores.

## Results

As indicated in the test case 3.3 setup description, a resolution of  $256^3$  DOF is expected to resolve almost all of the flow scales for a Reynolds number of 1600 and is thus very close to a DNS. We have conducted a series of simulations of this test case with varying number of elements and associated polynomial degree. Table 1 summarizes some selected setups and gives their computational effort in TauBench workunits. Note that the solution polynomial is of degree  $N$ , while the integration of the flux terms is achieved on a quadrature grid with  $M + 1$  nodes. Thus, Cases 1 and 2 are co-location DGSEM, while the rest use polynomial de-aliasing. Since the flow is essentially incompressible and thus the fluxes have quadratic non-linearities,  $M = \frac{3}{2}N$  results in a fully de-aliased solution. More details can be found e.g. in [1]. Cases 3 and 4 use a novel locally and temporally adaptive de-aliasing method.

Case	No. elements	N	DOF	M	No. of cores	Wall Time [s]	WU (Wall Time)	WU (CPU Time)
1	$16^3$	15	$256^3$	15	2048	4925	1044	2.14E6
2	$32^3$	7	$256^3$	7	4096	1430	303	1.24E6
3*	$32^3$	7	$256^3$	11	4096	8440	1788	7.33E6
4*	$48^3$	7	$384^3$	11	6912	27828	5898	4.07E7
5	$24^3$	7	$192^3$	11	6912	367	1044	2.54E6
6	$32^3$	7	$256^3$	11	4096	4380	928	3.80E6
7	$48^3$	7	$384^3$	11	6912	14880	3153	2.17E7
8	$16^3$	15	$256^3$	23	2048	21599	4556	9.33E6
9	$24^3$	15	$384^3$	23	6912	32554	6899	4.76E7

Table 1: Selected Taylor-Green vortex computations:  $N$ : polynomial degree;  $M$ : polynomial degree of fluxes; WU: Work units on CRAY XC40 “Hornet” with TauBench  $\approx 4.7$ s. \* indicates computation with adaptive polynomial de-aliasing.

Figure 2 shows the results for the enstrophy evolution for cases from Table 1. The left plot presents the results for the co-located DGSEM, the de-aliased and the adaptive de-aliased solution, all on a  $32^3$  grid with  $N = 7$ . The right plots shows the grid convergence for the fully de-aliased  $N = 7$  case.

Figure 3 shows a zoom-in on the dissipation rate and the grid convergence of the  $\epsilon_3$  dissipation rate. Figure 5 shows the isocontours of vorticity magnitude ( $\frac{L}{V_0}|\omega| = 10$ ) for the classical de-aliasing (left) and the adaptive approach (right).

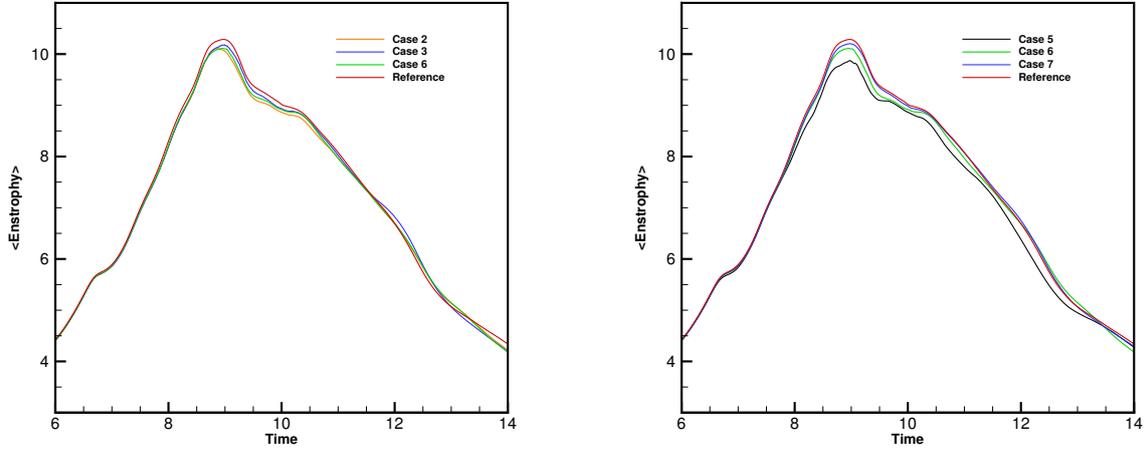


Figure 2: Evolution of enstrophy, *left*:  $32 \times 8$  solutions for collocation, polynomial de-aliasing and adaptive de-aliasing; *right*: Grid convergence for  $\mathcal{O}(8)$ ; refer to Tbl.1 for details.

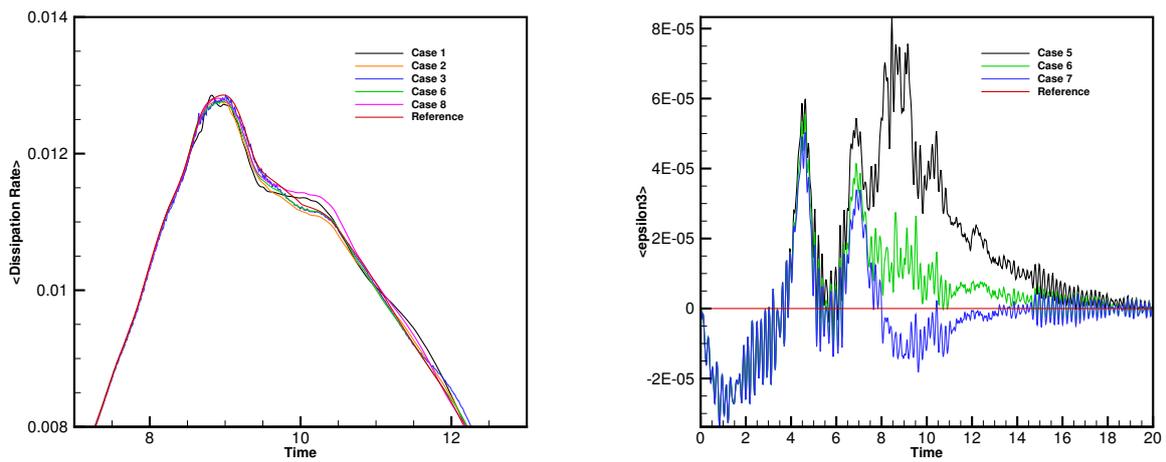


Figure 3: *Left*: Kinetic energy dissipation rate; *right*:  $\epsilon_3$  grid convergence; refer to Tbl.1 for details.

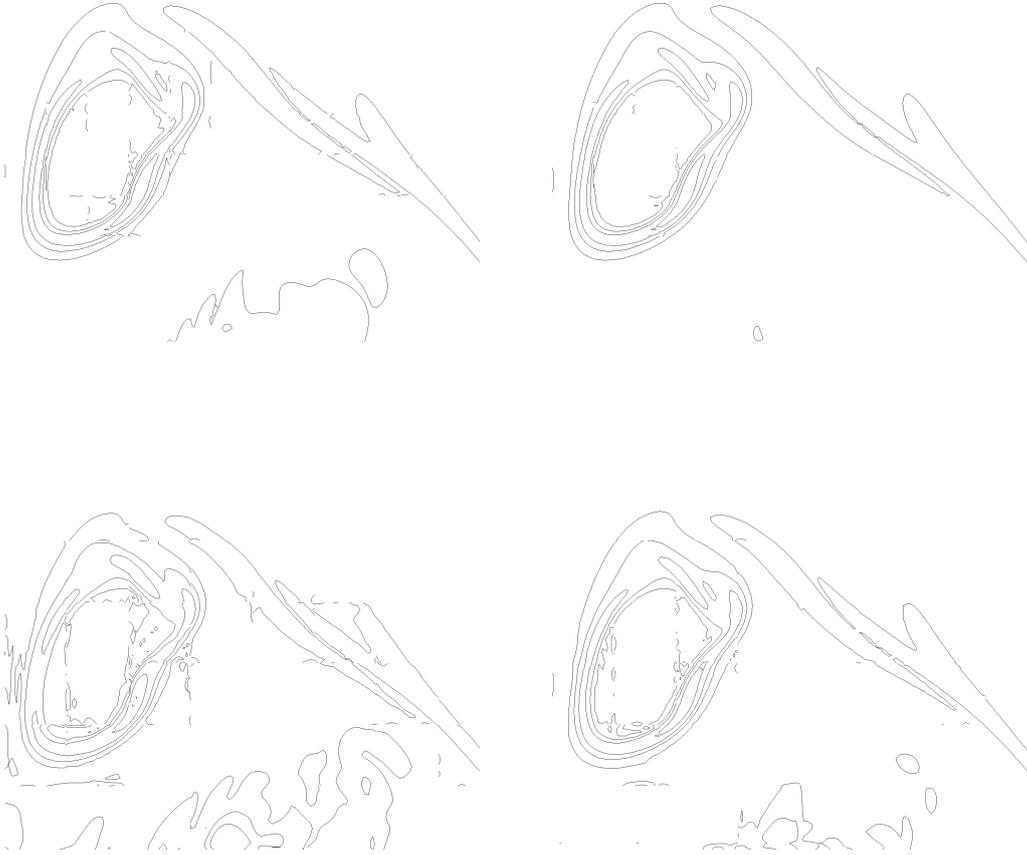


Figure 4: Vorticity contours; *top*: Cases 7 and 4; *bottom*: Cases 6 and 3

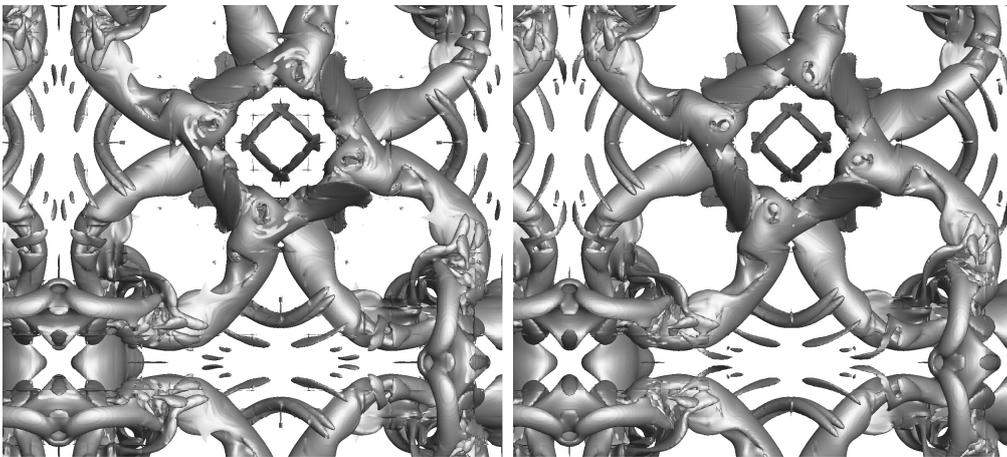


Figure 5: Isocontours of vorticity magnitude  $\frac{L}{V_0}|\omega| = 10$ : Cases 6 (left) and 3 (right).

## References

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